Exercise 19

Solve the steady-state surface wave problem (Debnath 1994, p. 47) on a running stream of infinite depth due to an external steady pressure applied to the free surface. The governing equation and the free surface conditions are

$$\phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty,\ -\infty < z < 0,\ t > 0,$$
$$\left.\begin{aligned} &\phi_x + U\phi_x + g\eta = -\frac{P}{\rho}\delta(x)\exp(\varepsilon t), \\ &\eta_t + U\eta_x = \phi_z \end{aligned}\right\} \quad \text{on } z = 0\ (\varepsilon > 0),$$
$$\phi_z \to 0 \quad \text{as } z \to -\infty.$$

where U is the stream velocity, $\phi(x, z, t)$ is the velocity potential, and $\eta(x, t)$ is the free surface elevation. [TYPO: This should be ϕ_t!]

Solution

In order for the first boundary condition to be dimensionally consistent, the first term must be ϕ_t, similar to the equation below it for η. The PDEs for ϕ and η are defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve them. We define the Fourier transform with respect to x here as

$$\mathcal{F}_x\{\phi(x, z, t)\} = \Phi(k, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\phi(x, z, t)\, dx,$$

which means the partial derivatives of ϕ with respect to x, z, and t transform as follows.

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial x^n}\right\} = (ik)^n \Phi(k, z, t)$$
$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial z^n}\right\} = \frac{d^n \Phi}{dz^n}$$
$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial t^n}\right\} = \frac{d^n \Phi}{dt^n}$$

Take the Fourier transform of both sides of the first PDE.

$$\mathcal{F}_x\{\phi_{xx} + \phi_{zz}\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_x\{\phi_{xx}\} + \mathcal{F}_x\{\phi_{zz}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Expand the coefficient of Φ.

$$-k^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Bring the term with Φ to the right side.

$$\frac{d^2\Phi}{dz^2} = k^2\Phi$$

We can write the solution to this ODE in terms of exponentials.

$$\Phi(k,z,t) = A(k,t)e^{|k|z} + B(k,t)e^{-|k|z}$$

We can use the last boundary condition here to figure out one of the constants. Taking the Fourier transform with respect to x of both sides of it gives us

$$\mathcal{F}_x\left\{\lim_{z\to-\infty}\frac{\partial\phi}{\partial z}\right\} = \mathcal{F}_x\{0\}.$$

Bring the transform inside the limit.

$$\lim_{z\to-\infty}\mathcal{F}_x\left\{\frac{\partial\phi}{\partial z}\right\} = 0$$

Transform the partial derivative.

$$\lim_{z\to-\infty}\frac{d\Phi}{dz} = 0$$

Differentiating Φ with respect to z, we obtain

$$\frac{d\Phi}{dz}(k,z,t) = A(k,t)|k|e^{|k|z} - B(k,t)|k|e^{-|k|z}.$$

In order for the boundary condition to be satisfied, we require that $B(k,t) = 0$.

$$\Phi(k,z,t) = A(k,t)e^{|k|z} \tag{1}$$

Take the Fourier transform with respect to x of the boundary conditions at $z = 0$ now.

$$\mathcal{F}_x\{\phi_t + U\phi_x + g\eta\} = \mathcal{F}_x\left\{-\frac{P}{\rho}\delta(x)e^{\varepsilon t}\right\}$$
$$\mathcal{F}_x\{\eta_t + U\eta_x\} = \mathcal{F}_x\{\phi_z\}$$

Use the linearity property.

$$\mathcal{F}_x\{\phi_t\} + U\mathcal{F}_x\{\phi_x\} + g\mathcal{F}_x\{\eta\} = -\frac{P}{\rho}e^{\varepsilon t}\mathcal{F}_x\left\{\delta(x)\right\}$$
$$\mathcal{F}_x\{\eta_t\} + U\mathcal{F}_x\{\eta_x\} = \mathcal{F}_x\{\phi_z\}$$

Transform the partial derivatives.

$$\frac{d\Phi}{dt} + U(ik)\Phi + gH = -\frac{P}{\rho\sqrt{2\pi}}e^{\varepsilon t} \tag{2}$$
$$\frac{dH}{dt} + U(ik)H = \frac{d\Phi}{dz} \tag{3}$$

Solve equation (2) for H.

$$H(k,t) = -\frac{1}{g}\left(\frac{P}{\rho\sqrt{2\pi}}e^{\varepsilon t} + Uik\Phi + \frac{d\Phi}{dt}\right)$$

Take a derivative of this with respect to t.

$$\frac{dH}{dt} = -\frac{1}{g}\left(\frac{P\varepsilon}{\rho\sqrt{2\pi}}e^{\varepsilon t} + Uik\frac{d\Phi}{dt} + \frac{d^2\Phi}{dt^2}\right)$$

Use equation (1) to write expressions for $d\Phi/dt$ and $d^2\Phi/dt^2$.

$$\frac{d\Phi}{dt} = \frac{dA}{dt}e^{|k|z} \quad \rightarrow \quad \left.\frac{d\Phi}{dt}\right|_{z=0} = \frac{dA}{dt}$$
$$\frac{d^2\Phi}{dt^2} = \frac{d^2A}{dt^2}e^{|k|z} \quad \rightarrow \quad \left.\frac{d^2\Phi}{dt^2}\right|_{z=0} = \frac{d^2A}{dt^2}$$

The equations for H and dH/dt become (noting that $\Phi(k,0,t) = A(k,t)$)

$$H(k,t) = -\frac{1}{g}\left(\frac{P}{\rho\sqrt{2\pi}}e^{\varepsilon t} + UikA + \frac{dA}{dt}\right)$$
$$\frac{dH}{dt} = -\frac{1}{g}\left(\frac{P\varepsilon}{\rho\sqrt{2\pi}}e^{\varepsilon t} + Uik\frac{dA}{dt} + \frac{d^2A}{dt^2}\right)$$

Plug these two equations into equation (3) to get an ODE for $A(k,t)$. The right side is obtained by differentiating equation (1) with respect to z and then setting z equal to zero.

$$-\frac{1}{g}\left(\frac{P\varepsilon}{\rho\sqrt{2\pi}}e^{\varepsilon t} + Uik\frac{dA}{dt} + \frac{d^2A}{dt^2}\right) - \frac{Uik}{g}\left(\frac{P}{\rho\sqrt{2\pi}}e^{\varepsilon t} + UikA + \frac{dA}{dt}\right) = |k|A$$

Multiply both sides by $-g$, expand the left side, and combine like-terms.

$$\frac{d^2A}{dt^2} + 2Uik\frac{dA}{dt} - k^2U^2A + \frac{P(Uik+\varepsilon)}{\rho\sqrt{2\pi}}e^{\varepsilon t} = -g|k|A$$

Bring the term with $e^{\varepsilon t}$ to the right side and bring $g|k|A$ to the left.

$$\frac{d^2A}{dt^2} + 2Uik\frac{dA}{dt} + (g|k| - k^2U^2)A = -\frac{P(Uik+\varepsilon)}{\rho\sqrt{2\pi}}e^{\varepsilon t}$$

Since we only care about the steady state, we only need a particular solution to this ODE. The inhomogeneous term is $e^{\varepsilon t}$, so the particular solution is of the form $ce^{\epsilon t}$. We determine the constant c by plugging this form into the ODE and solving the resulting equation for it. Doing so yields

$$A(k,t) = -\frac{P(Uik+\varepsilon)}{\rho\sqrt{2\pi}[g|k| - (kU - i\varepsilon)^2]}e^{\varepsilon t},$$

so from equation (1) we know what $\Phi(k,z,t)$ is. Factor i from the numerator.

$$\Phi(k,z,t) = \frac{iPe^{\varepsilon t}}{\rho\sqrt{2\pi}}\frac{Uk - i\varepsilon}{(Uk-i\varepsilon)^2 - g|k|}e^{|k|z} \tag{4}$$

To obtain $\phi(x,z,t)$, take the inverse Fourier transform of $\Phi(k,z,t)$. It is defined as

$$\mathcal{F}^{-1}\{\Phi(k,z,t)\} = \phi(x,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k,z,t)e^{ikx}\,dk.$$

Therefore,

$$\boxed{\phi(x,z,t) = \frac{iPe^{\varepsilon t}}{2\pi\rho}\int_{-\infty}^{\infty}\frac{Uk - i\varepsilon}{(Uk - i\varepsilon)^2 - g|k|}e^{|k|z+ikx}\,dk.}$$

To solve for H, use equation (3). Differentiate equation (4) with respect to z and set z equal to zero to obtain the right side.

$$\frac{dH}{dt} + UikH = \frac{iP|k|}{\rho\sqrt{2\pi}}\frac{Uk - i\varepsilon}{(Uk - i\varepsilon)^2 - g|k|}e^{\varepsilon t}$$

As explained before, we only care about the steady state, so we only need a particular solution of this equation. The inhomogeneous term is $e^{\varepsilon t}$, so the particular solution will have the form $de^{\varepsilon t}$. Substitute this form into the ODE for H to get an equation for the constant d. The particular solution is

$$H(k,t) = -\frac{P|k|}{\sqrt{2\pi}\rho[g|k| - (Uk - ik)^2]}e^{\varepsilon t}.$$

Now that we have $H(k,t)$, we can get $\eta(x,t)$ by taking the inverse Fourier transform of it. Therefore,

$$\boxed{\eta(x,t) = \frac{Pe^{\varepsilon t}}{2\pi\rho}\int_{-\infty}^{\infty}\frac{|k|e^{ikx}}{(Uk - i\varepsilon)^2 - g|k|}\,dk.}$$